On Equilibrium Shapes of Solid Solar System Bodies. K.A. Holsapple, Aeronautics and Astronautics, University of Washington, 352400, Seattle, WA 98195; holsapple@aa.washington.edu.

The determination of the possible equilibrium shapes for celestial bodies with gravitational, rotational and/or tidal forces is a classical problem that has spanned over three centuries. Newton in 1687, Maclaurin in 1742 (Maclaurin spheroids), Jacobi in 1834 (Jacobi Ellipsoids), Roche in 1850 (the Roche limit) and Poincaré in 1885 are classical references, as presented by Chandrasekhar [1]. All of these results are for an incompressible fluid body, which is unlikely to be a good representation of most solar system bodies. Later, others considered linear elastic bodies. Some approaches looked only at forces across entire cross sections, while others derive the complete stress field for various but specified combinations of shape (spherical or ellipsoidal), and various combinations of the three force types [2-8]. Then they compared the stress results to criteria for either yield or fracture. These equilibrium solutions are obtainable for any initial shape, and there is no limit on possible shapes until one imposes some yield or fracture criteria.

These elastic approaches have two major deficiencies. First, while the determination of the elastic deformation from an initial state has a unique solution, the choice of that initial state is not unique. Implicit in any elastic analyses is some assumption about the initial state. Usually, it is that the initial state is stress-free: there are no "residual" stresses with zero loads. That is also not likely to be a good assumption for solar-system bodies. The present state of those bodies is a culmination of a complicated past history, possibly involving collisions, disruption, accumulation and large scale yielding and reshaping. Such processes would create an underlying residual stress field that cannot be known.

The second problem with the elastic analyses is that, when the stresses are found that conform to some yield criteria, there is no information of the deformation that would arise. Does the body disrupt? Or does it simply inelastically deform to some new equilibrium configuration?

In principal, a complete analysis using an appropriate inelastic theory could follow any assumed history, from any assumed initial state, to determine the present shape and stress state. Then also the deformation rules of the theory (flow rules for plasticity) would determine the nature of the inelastic deformation and possible reshaping (deformation to a new equilibrium state) or disruption when yielding is indicated. But, there is still the uncertainty about the initial state, so that approach is also unsatisfactory.

However, all is not lost: there is a powerful approach in plasticity theories that can be used to determine the limit loads: those that lead to unconstrained plastic flow. Those limit loads are independent of any assumed initial state, or of any residual stress state that might exist at any one time with any one loading history. That is really the information that is wanted in any case.

Then, here is presented a summary of a recently completed study, Holsapple [9], in the same spirit as for the fluid bodies: limits on possible equilibrium shapes are sought, but for solid bodies. Specific results are obtained for a cohesionless elastic-plastic general ellipsoidal solid body (axes lengths a,b,c) governed by the Mohr-Coloumb yield criteria, using limit-load analyses. That Mohr-Coloumb model is used commonly in soil mechanics, and is thought to be particularly appropriate for a rubble-pile re-accumulated structure of asteroids. It is characterized by a single material property, the angle of friction \(\phi\).

Since a fluid is the special case where that angle is zero, all of the classical results (Maclaurin spheroids, Jacobi ellipsoids, Roche limits) for fluids are included as special cases in the results. Thus, unique limit equilibrium stress fields and shapes are found without having to analyze entire past histories. The results show that there exists a region of permissible combinations of aspect ratios and spin rates, centered about the special equilibrium fluid states of Maclaurin and Jacobi.

The analysis begins with looking for equilibrium stress fields of a quadratic form:

\[
\begin{align*}
\sigma_x &= k_1 + k_2 x^2 + k_3 y^2 + k_4 z^2 \\
\sigma_y &= k_5 + k_6 x^2 + k_7 y^2 + k_8 z^2 \\
\sigma_z &= k_9 + k_{10} x^2 + k_{11} y^2 + k_{12} z^2 \\
\tau_{xy} &= k_{13} x y \\
\tau_{xz} &= k_{14} x z \\
\tau_{yz} &= k_{15} y z
\end{align*}
\]

Imposition of the three equations of stress equilibrium and the boundary conditions of zero tractions allows 12 of the 15 unknown constants to be determined. The remaining three are determined by assumptions about the constitutive equations.

If the elastic solution from an initial stress-free state is desired, the equations of strain compatibility are used to determine those 3 remaining constants, and the results of Dobrovolskis[7] are obtained, except that by using MATHEMATIC, the results are obtainable in entirely closed-form algebraic form.

To get the elastic-plastic solutions, the Mohr-Coloumb yield conditions are used to get the final three constants. The resulting stresses are very simple in form: the shear stresses are all zero and the normal stresses are given as
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\[ \sigma_x = -pk, a^2 \left[ 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 \right] \]
\[ \sigma_y = -pk, b^2 \left[ 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 \right] \]
\[ \sigma_z = -pk, c^2 \left[ 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 \right] \]

where the \( k_x, k_y, k_z \) are the components of the quadratic body force potential function for any of gravitational, rotational or tidal.

Since the ratios of the stresses are constant throughout the body, these solutions have Mohr's circles that are tangent to a Mohr-Coloumb criteria, with a single specific friction angle, at all points of the body. If that angle of friction is less than that of the actual body, those solutions are only one of many equilibrium elastic-plastic solutions and are definitely not unique. However, if they are all tangent to the actual angle of friction criteria, then they give the unique limit load for unconstrained plastic flow of the body.

Thus, one can set the required angle of friction equal to any given value, and solve for the corresponding limit loads. Those limit loads are expressible as a scaled spin rate \( \Omega = \omega / \sqrt{\rho G} \) depending on three dimensionless parameters: two shape factors of geometry \( \alpha = c/a, \beta = b/a \), and the angle of friction.

To make plots, contours of required friction angle can be determined on a two-dimensional plot using \( \alpha \) and \( \Omega \), if one assumes some particular value or relation for \( \beta \). Then actual data for specific bodies can be located on that plot and the required friction angle determined.

The following three figures, from [9], assume a prolate spheroidal shape and shows the configurations of large sets of C-type, S-type and M-type asteroids, from the database of Pravec and Harris [10]. It is observed that almost all are well within the possible equilibrium regions with a very modest angle of friction: all of these could exist with a cohesionless rubble-pile structure.

Other specific results will be presented.

References: